Chiral algebras

X = smooth curve

$$X \overset{\Delta}{\hookrightarrow} X^2 \overset{j}{\hookleftarrow} X^2 \setminus \Delta$$

$M_1, M_2, N \in D\text{-mod}(X)$

$\mathrm{Hom}^{ch}(M_1, M_2; N)$
$:= \mathrm{Hom}_{D\text{-mod}(X^2)}(j_* j^*(M_1 \boxtimes M_2), \Delta_{*,dR} N)$

$\mathrm{Hom}^{ch}(M_1, \cdots, M_n; N)$ similar

"pseudo-tensor structure" $\overset{ch}{\otimes}$

A <u>nonunital chiral algebra</u> on X is a Lie algebra in $(D\text{-mod}(X), \overset{ch}{\otimes})$.

$A \in D\text{-mod}(X)$, $j_* j^*(A \boxtimes A) \to \Delta_{*,dR} A$
+ skew-symmetry, Jacobi identity

<u>Example</u> $A = \omega_X[-1]$

$$\begin{array}{c} \Delta_{*,dR} \Delta^! \omega_{X^2} \to \omega_{X^2} \to j_* j^* \omega_{X^2} \\ \parallel \\ \Delta_{*,dR}\,\omega_X \quad \rightsquigarrow j_* j^* \omega_{X^2} \to \Delta_{*,dR}\,\omega_X[1] \end{array}$$

$\rightsquigarrow j_* j^*(\omega_X[-1] \boxtimes \omega_X[-1]) = j_* j^* \omega_{X^2}[-2]$
$\qquad \to \Delta_{*,dR}\, \omega_X[-1]$

antisymmetry follows from $\tau^! \omega_{X^2} \cong \omega_{X^2}$

Jacobi identity follows from exactness of Cousin complex on X^3

A <u>(unital) chiral algebra</u> is a nonunital chiral algebra A equipped w/ a morphism of chiral algebras $\omega_X[-1] \to A$ s.t.

$$j_* j^*(\omega_X[-1] \boxtimes A) \to j_* j^*(A \boxtimes A) \to \Delta_{*,dR} A$$

equals the boundary map in

$$\begin{array}{l} \Delta_{*,dR} \Delta^!(\omega_X \boxtimes A) \to \omega_X \boxtimes A \to j_* j^*(\omega_X \boxtimes A) \\ \quad\ \| \\ \Delta_{*,dR} A \end{array}$$

Vertex algebras $\rightsquigarrow$ chiral algebras

$\underline{\mathrm{Aut}}(\mathcal{O}) :=$ group ind-scheme of cts automorphisms of $\mathcal{O} = \mathbb{C}[[z]]$

$$\mathrm{Map}(\mathrm{Spec}\, R, \underline{\mathrm{Aut}}(\mathcal{O})) = \mathrm{Aut}^{\mathrm{cts}}_{R\text{-alg}}(R[[z]])$$

$$\mathrm{Lie}(\underline{\mathrm{Aut}}(\mathcal{O})) = \mathrm{Der}(\mathcal{O}) \quad \text{(cts derivations)}$$
$$= \mathcal{O} \cdot \partial_z$$

$L_n = -z^{n+1}\partial_z$, $n \geq -1$ topological basis for $\mathrm{Der}(\mathcal{O})$

<u>Def'n</u> A <u>quasi-conformal</u> structure on a vertex algebra V is an action of $\mathrm{Der}(\mathcal{O})$ on the vector space V s.t.

i) the action exponentiates to $\underline{\mathrm{Aut}}(\mathcal{O}) \curvearrowright V$,
ii) $L_{-1} = -\partial_z$
iii) $\forall\, n \geq -1$, $A \in V$, we have

$$[L_n, Y(A,w)] = \sum_{m \geq -1} \frac{1}{(m+1)!} \left(\partial_w^{m+1} w^{n+1} Y(L_m A, w) \right).$$

(i) $\Longleftrightarrow$ (ia) + (ib)

ia) $L_0 = -z\partial_z$ acts semisimply w/ integral eigenvalues

ib) $\mathrm{Der}_0(\mathcal{O}) := z\mathbb{C}[[z]]\partial_z$ acts locally nilpotently

<u>Example</u> The Kac-Moody vertex algebra $V_\kappa(\mathfrak{g})$ has a natural quasi-conformal structure (F-BZ Example 6.4.9).

X as before

$$P_X = \{(x,z) \mid x \in X,\ z \in \hat{\mathcal{O}}_{X,x} \text{ coordinate}\}$$
$$\downarrow$$
X torsor for $\underline{\mathrm{Aut}}_0(\mathcal{O})$, Zariski-locally trivial

$$\mathrm{Map}(\mathrm{Spec}\, R, \underline{\mathrm{Aut}}_0(\mathcal{O})) = \{\varphi \in \mathrm{Aut}^{\text{cts}}_{R\text{-alg}}(R[[z]]) \mid \varphi(z)(0) = 0\}$$

$$\mathrm{Der}_0(\mathcal{O}) = \mathrm{Lie}(\underline{\mathrm{Aut}}_0(\mathcal{O}))$$

V = quasi-conformal vertex algebra

$$\rightsquigarrow \mathcal{V} := V \overset{\underline{\mathrm{Aut}}_+(\mathcal{O})}{\times} P_X \in \mathrm{QCoh}(X)$$
locally free

Observation: $\underline{Aut}(\mathcal{O}) \circlearrowright P_X$ and

$$P_X / \underline{Aut}(\mathcal{O}) \cong X_{dR},$$

where

$$Map(Spec\, R, X_{dR}) = Map(R_{red}, X).$$

$$\rightsquigarrow V \overset{\underline{Aut}(\mathcal{O})}{\underset{\times}{}} P_X \in QCoh(X_{dR})$$

Grothendieck:

$$QCoh(X_{dR}) \cong D\text{-}mod(X)$$

$\Rightarrow T = L_{-1} \circlearrowright V \rightsquigarrow$ flat connection on $\mathcal{V}$

So far we have constructed
$\mathcal{V} \in D\text{-}mod(X)$.

Next, we will construct a chiral Lie bracket

$$j_* j^* (\mathcal{V} \boxtimes \mathcal{V}) \to \Delta_{dR,*} \mathcal{V}$$

in $D\text{-}mod(X)$.

Let's consider the underlying quasicoherent sheaves (using left $\mathcal{D}$-modules), and restrict to $(\hat{X}_x)^2$.

Fixing a coordinate at x, we have

$$j_*j^*(\mathcal{V}\boxtimes\mathcal{V})\big|_{(\hat{X}_x)^2} \cong (V\otimes V)[[z,w]][(z-w)^{-1}]$$

$$\Delta_{dR,*}(\mathcal{V})\big|_{(\hat{X}_x)^2} \cong V[[z,w]][(z-w)^{-1}]/V[[z,w]]$$

<u>Claim</u> (F-BZ Ch. 19) $\exists!$ chiral bracket on $\mathcal{V}$ given by

$$A\boxtimes B \mapsto Y(A,z-w)\cdot B \ (\mathrm{mod}\ V[[z,w]])$$

over $(\hat{X}_x)^2$.

Unit: $\mathbb{C}|0\rangle \to V \rightsquigarrow \omega_X[-1] \to \mathcal{V}$

Not all chiral algebras arise in this way. Those that do are <u>universal</u> in the sense that $\forall$ étale $f: X\to Y$, we have

$$f^!\mathcal{V}_Y \cong \mathcal{V}_X.$$

Factorization algebras

Ran := moduli space of nonempty finite subsets of X

$\cong \operatorname*{colim}_{\substack{\#I<\infty \\ I\neq\emptyset}} X^I$

$$(\mathrm{Ran}\times\mathrm{Ran})_{disj} \overset{j}{\hookrightarrow} \mathrm{Ran}\times\mathrm{Ran} \xrightarrow{\text{union}} \mathrm{Ran}$$

$M_1, M_2 \in D\text{-mod}(\mathrm{Ran})$

$\rightsquigarrow M_1 \overset{ch}{\otimes} M_2 := \mathrm{union}_{dR,*}\, j_* j^* (M_1 \boxtimes M_2)$

non-unital symm. mon. structure on $D\text{-mod}(\mathrm{Ran})$

NB: non-unital chiral algebra on X = Lie algebra in $(D\text{-mod}(\mathrm{Ran}), \overset{ch}{\otimes})$ of the form $i_{dR,*}A$, where $i : X \to \mathrm{Ran}$.

__Def'n__ A __non-unital factorization algebra__ on X is a non-unital cocommutative coalgebra in $(D\text{-mod}(\mathrm{Ran}), \overset{ch}{\otimes})$ s.t.

$$A \to A \overset{ch}{\otimes} A = \mathrm{union}_{dR,*}\, j_* j^* (A \boxtimes A)$$

$\rightsquigarrow j^* \mathrm{union}^! A \xrightarrow{\sim} j^*(A \boxtimes A).$

$$\mathrm{LieAlg}^{\otimes^{ch}}(\text{D-mod}(\mathrm{Ran})) \to \mathrm{CocomCoalg}^{\otimes^{ch}}(\text{D-mod}(\mathrm{Ran}))$$
$$A \mapsto C_+^{ch}(A)$$

<u>Theorem</u> (Francis-Gaitsgory) This is an equivalence, and restricts to

$$\left\{ \begin{array}{c} \text{non-unital chiral} \\ \text{algebras} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{non-unital fact.} \\ \text{algebras} \end{array} \right\}.$$

Many factorization algebras admit geometric constructions

$Y \xrightarrow{\pi} \mathrm{Ran}_{dR}$ non-unital fact. space

$$(\mathrm{Ran}_{dR} \times \mathrm{Ran}_{dR})_{disj} \underset{\mathrm{Ran}_{dR}}{\times} Y$$
$$\downarrow \wr$$
$$(\mathrm{Ran}_{dR} \times \mathrm{Ran}_{dR})_{disj} \underset{\mathrm{Ran}_{dR} \times \mathrm{Ran}_{dR}}{\times} (Y \times Y)$$

associative + commutative

Assume Y locally of finite type

$\rightsquigarrow \mathrm{Dist}(Y) := \pi_*^{\mathrm{IndCoh}} \pi^! \omega_Y$ fact. alg.

E.g., $Y = \widehat{Gr}_{G, Ran}$

$\rightsquigarrow$ $Dist(Y)$ = fact. alg. corresponding to the vertex/chiral algebra $V_0(\mathfrak{g})$

Feigin-Frenkel at critical level

$V =$ vertex algebra

$$Z(V) := \{ B \in V \mid A_{(n)} B = 0 \ \forall A \in V, n \geq 0 \}$$
$$= \{ B \in V \mid [Y(A,z), Y(B,w)] = 0 \ \forall A \in V \}$$

One checks that $Z(V)$ is a comm. vertex subalgebra of V.

$$\kappa_{crit} := -\tfrac{1}{2} \kappa_{Kil}$$

Proposition $\kappa \neq \kappa_{crit}$
$$\Rightarrow Z(V_\kappa(\mathfrak{g})) = \mathbb{C}|0\rangle.$$

One checks that

$$Z(V_\kappa(\mathfrak{g})) \cong V_\kappa(\mathfrak{g})^{\hat{\mathfrak{g}}_\kappa} \cong \mathrm{End}_{\hat{\mathfrak{g}}_\kappa}(V_\kappa(\mathfrak{g}))$$

as associative algebras.

Recall the functor

$$DS : \hat{\mathfrak{g}}_\kappa\text{-mod} \longrightarrow \mathrm{Vect}$$

$$Z(V_\kappa(\mathfrak{g})) \hookrightarrow DS(V_\kappa(\mathfrak{g})) =: W_{\mathfrak{g},\kappa}$$

$|0\rangle \in DS(V_\kappa(\mathfrak{g}))$

$$\rightsquigarrow Z(V_\kappa(\mathfrak{g})) \longrightarrow DS(V_\kappa(\mathfrak{g})) \quad (*)$$

<u>Theorem</u> (F-F) $\kappa = \kappa_{crit} \Rightarrow (*)$ is an isomorphism of quasi-conformal (comm.) vertex algebras.

<u>Corollary</u> $Z(V_{\kappa_{crit}}(\mathfrak{g})) \cong \mathrm{Fun}(Op_{\check{G}}(D))$ as quasi-conformal vertex algebras.

<u>Proof</u> $\check{\kappa}_{crit} = \infty$

$$\Rightarrow W_{\mathfrak{g},\kappa_{crit}} \overset{\text{F-F}}{\cong} W_{\mathfrak{g},\infty} \overset{\text{last time}}{\cong} \mathrm{Fun}(Op_{\check{G}}(D)). \quad \square$$